Weak solutions of the three-dimensional vorticity equation with vortex singularities

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The use of a modified scheme for the dynamics of vortex singularities is shown to lead to a weak solution of the three-dimensional inviscid incompressible vorticity equation.

Two-dimensional point vortices have been shown to be a weak solution of the vorticity equation[1] and have proved useful in the investigation of vortical flows of an incompressible fluid. The concept of vortex singularities has been extended to three dimensions, leading to the method of vortex sticks, also called *vortons*. In this case, the vorticity is written as

$$\boldsymbol{\omega}(\mathbf{x},t) = \sum_k \boldsymbol{\alpha}_k(t)\, \delta(\mathbf{x} - \mathbf{x}_k(t)) = \sum_k \boldsymbol{\omega}_k . \tag{1}$$

This vorticity field cannot be the limit of a sequence of smooth solutions, in contrast to point vortices in two dimensions. This is because the vorticity field (1) is not divergence-free. The velocity field is given by the Biot–Savart integral on the vortex singularities plus an external irrotational field

$$\mathbf{u}(\mathbf{x},t) = \left(\frac{1}{4\pi}\right)\sum_k \frac{\boldsymbol{\alpha}_k \wedge (\mathbf{x} - \mathbf{x}_k)}{\|\mathbf{x} - \mathbf{x}_k\|^3} + \nabla\phi(\mathbf{x},t)$$

$$= \left(\frac{1}{4\pi}\right)\frac{\boldsymbol{\alpha}_k \wedge (\mathbf{x} - \mathbf{x}_k)}{\|\mathbf{x} - \mathbf{x}_k\|^3} + \mathbf{u}_k(\mathbf{x},t) , \tag{2}$$

where $\mathbf{u}_k(\mathbf{x},t)$ is the velocity field without the contribution of the k vorton. The evolution equations for the vortons are usually taken as

$$\frac{d}{dt}\mathbf{x}_k = \mathbf{u}_k(\mathbf{x}_k,t) , \tag{3}$$

$$\frac{d}{dt}\boldsymbol{\alpha}_k = (\boldsymbol{\alpha}_k\cdot\nabla)\mathbf{u}_k(\mathbf{x}_k,t) . \tag{4}$$

Saffman and Meiron[1] have shown that the formulation (3) and (4) does not constitute a weak solution of the three-dimensional vorticity equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} . \tag{5}$$

However, as noticed by Rehbach[2] (see also Cantaloube and Huberson[3]), Eq. (5) is equivalent to

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla^T)\mathbf{u} \tag{6a}$$

or

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = \frac{1}{2}[\boldsymbol{\omega}\cdot(\nabla + \nabla^T)]\mathbf{u} . \tag{6b}$$

This is so because $[\boldsymbol{\omega}\cdot(\nabla - \nabla^T)]\mathbf{u} = (\nabla\wedge\mathbf{u})\wedge\boldsymbol{\omega} = \boldsymbol{\omega}\wedge\boldsymbol{\omega} = 0$. The formulation (6a) suggests the evolution equation

$$\frac{d}{dt}\boldsymbol{\alpha}_k = (\boldsymbol{\alpha}_k\cdot\nabla^T)\mathbf{u}_k(\mathbf{x}_k,t) , \tag{7}$$

while the formulation (6b) suggests the evolution equation

$$\frac{d}{dt}\boldsymbol{\alpha}_k = \frac{1}{2}[\boldsymbol{\alpha}_k\cdot(\nabla + \nabla^T)]\mathbf{u}_k(\mathbf{x}_k,t) , \tag{8}$$

a form favored by Rehbach because the symmetry of the matrix $(\nabla + \nabla^T)\mathbf{u}$ yields computational savings.

As indicated by Cottet,[4,5] the choice of (7) leads to the conservation of total vorticity, a property not satisfied by the usual formulation (4) or Rehbach formulation (8). Actually, total vorticity is only conserved in the absence of an external potential flow as pointed out by Saffman.[6] More specifically,

$$\frac{d}{dt}\left(\sum_k \alpha_{k_i}(t)\right) = \sum_k \alpha_{k_j}(t)\,\frac{\partial^2}{\partial x_i\,\partial x_j}\phi(\mathbf{x}_k,t) . \tag{9}$$

The formulations (4), (7), and (8) would be equivalent if the vorticity field as defined by (1) was equal to the curl of the velocity field (2). Unfortunately, this is not the case as a result of the nonzero divergence of the field (1). There is thus a fundamental difference between the formulations (4), (7), and (8) for the dynamics of vortex singularities.

The fact that the formulation (7) conserves total vorticity suggests that it might be a weak solution of the vorticity equation. The purpose of this Letter is to show that indeed it is. The proof is done using the same procedure as in Saffman and Meiron[1] but with the evolution equations (3) and (7).

First, since $\mathbf{u}_k(\mathbf{x},t)$ is smooth in the neighborhood of $\mathbf{x}_k$, (3) and (7) imply that

$$\frac{\partial}{\partial t}\boldsymbol{\omega}_k + (\mathbf{u}_k\cdot\nabla)\boldsymbol{\omega}_k - (\boldsymbol{\omega}_k\cdot\nabla^T)\mathbf{u}_k = 0 \tag{10}$$

at all points including $\mathbf{x} = \mathbf{x}_k$. It remains to consider whether

$$[(\mathbf{u} - \mathbf{u}_k)\cdot\nabla]\boldsymbol{\omega}_k - (\boldsymbol{\omega}_k\cdot\nabla^T)(\mathbf{u} - \mathbf{u}_k) = 0 \tag{11}$$

at $\mathbf{x}_k$ in a weak sense. We take a local coordinate system with the origin at $\mathbf{x}_k$ and use a suffix notation. The suffix k of the vorton is dropped. It thus remains to consider whether the following integral vanishes for an arbitrary smooth function $f(\mathbf{x})$:

$$\int f(\mathbf{x})\left[\epsilon_{pjk}\frac{\alpha_j x_k}{r^3}\frac{\partial}{\partial x_p}(\alpha_i\delta(\mathbf{x})) - \delta(\mathbf{x})\alpha_p\frac{\partial}{\partial x_i}\left(\epsilon_{pjk}\frac{\alpha_j x_k}{r^3}\right)\right]d\mathbf{x} . \tag{12}$$

The first term is integrated by parts and (12) becomes

$$-\int \delta(\mathbf{x})\left[\epsilon_{pjk}\alpha_i\alpha_j \frac{\partial}{\partial x_p}\left(\frac{f x_k}{r^3}\right) + \epsilon_{pjk}\alpha_j\alpha_p f \frac{\partial}{\partial x_i}\left(\frac{x_k}{r^3}\right)\right] d\mathbf{x}\,. \tag{13}$$

The second term in (13) is different from the one obtained using the evolution equations (3) and (4),[1] namely,

$$-\int \delta(\mathbf{x})\left[\epsilon_{pjk}\alpha_i\alpha_j \frac{\partial}{\partial x_p}\left(\frac{f x_k}{r^3}\right) + \epsilon_{ijk}\alpha_j\alpha_p f \frac{\partial}{\partial x_p}\left(\frac{x_k}{r^3}\right)\right] d\mathbf{x}. \tag{14}$$

As pointed out by Greengard and Thomann,[7] one needs to assume a radially symmetric regularization of the δ function for the integrals above, interpreted in the principal value sense, to be well-defined. Such a regularization is to be understood in the present context.

Being smooth, $f(\mathbf{x})$ has a Taylor series

$$f(\mathbf{x}) = f(0) + x_q f_q(0) + \tfrac{1}{2} x_q x_r f_{qr}(0) + \cdots. \tag{15}$$

From the symmetry of the integrand in (13), it follows that only even powers of the coordinates need to be considered. Moreover, terms of order 4 and higher vanish sufficiently fast so as to cancel the delta function contribution. Thus only the constant term and the quadratic terms need to be considered. The first part of (13) [or (14)] gives

$$\begin{aligned} & f(0)\left(\epsilon_{kjk}\alpha_i\alpha_j \int \frac{\delta(\mathbf{x})}{r^3}\, d\mathbf{x} - 3\epsilon_{pjk}\alpha_i\alpha_j \int \frac{\delta(\mathbf{x})}{r^3} x_k x_p\, d\mathbf{x}\right) \\ & \quad + f_{qr}(0)\left(\epsilon_{kjk}\alpha_i\alpha_j \int \frac{\delta(\mathbf{x})}{r^3} x_q x_r\, d\mathbf{x}\right. \\ & \quad + \epsilon_{qjk}\alpha_i\alpha_j \int \frac{\delta(\mathbf{x})}{r^3} x_r x_k\, d\mathbf{x} \\ & \quad + \epsilon_{rjk}\alpha_i\alpha_j \int \frac{\delta(\mathbf{x})}{r^3} x_q x_k\, d\mathbf{x} \\ & \quad \left. - 3\epsilon_{pjk}\alpha_i\alpha_j \int \frac{\delta(\mathbf{x})}{r^3} x_q x_r x_k x_p\, d\mathbf{x}\right). \end{aligned} \tag{16}$$

The first and third terms in (16) are zero since $\epsilon_{kjk} = 0$. The second term vanishes since $\epsilon_{pjk} x_k x_p = 0$. The fourth and fifth terms are identical since $f_{qr}(0) = f_{rq}(0)$. They are essentially proportional to

$$f_{qr}(0)\epsilon_{qjk}\alpha_i\alpha_j\delta_{rk} = f_{qr}(0)\epsilon_{qjr}\alpha_i\alpha_j = 0\,. \tag{17}$$

The last term is proportional to

$$f_{qr}(0)\epsilon_{pjk}\alpha_i\alpha_j \int x_q x_r x_k x_p\, dS\,. \tag{18}$$

Using the result that

$$\int x_i x_j x_k x_l\, dS \propto \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\,, \tag{19}$$

it is easily seen that this term also vanishes. We thus have that the first term of (13) [or (14)] vanishes. The term that does not vanish in (14) is the second term. It is proportional to[1]

$$f_{qr}(0)(\epsilon_{ijq}\alpha_j\alpha_r + \epsilon_{ijr}\alpha_j\alpha_q)\,, \tag{20}$$

thus showing that the usual vorton formulation is not a weak solution in three dimensions.

With the new formulation, the second term in (13) vanishes trivially even before integration since $\epsilon_{pjk}\alpha_j\alpha_p = 0$. We thus have a weak solution of the vorticity equation in three dimensions.

The modified formulation for the dynamics of vortex singularities in three dimensions is a weak solution of the vorticity equation. That property and the conservation of total vorticity leads us to believe that this formulation is more suited than the usual one to the representation of three-dimensional vortical flows with a limited number of vortex singularities.

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